Geometry of q-normal distribution through non-Fisher metric

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Tsallis entropy $\frac{1}{1-q}\left\{\int \mathrm{d}x\, p_q(x)^q - 1\right\}$

Given two independent systems A and B with Tsallis entropy S_q^A and S_q^B respectively.

Let Tsallis entropy of the compound system $A \cup B$ be $S_q^{A\cup B}$, then

$$S_q^{A\cup B} = S_q^A + S_q^B + (1-q)S_q^A S_q^B$$

➡ **Non-additive entropy**

Boltzmann-Shannon entropy（q = 1）is the special case,that is additive.

The q-normal distribution is obtained by the following variation problem

Tsallis entropy

The 1st q-moment

$$\frac{1}{1-q}\left\{\int \mathrm{d}x\, p(x)^q - 1\right\}$$
$$+ A\left\{\int \mathrm{d}x\, p(x) - 1\right\} + B\left\{\frac{1}{D}\int \mathrm{d}x\, p(x)^q\, x - \mu\right\}$$
$$+ C\left\{\frac{1}{D}\int \mathrm{d}x\, p(x)^q\,(x-\mu)^2 - \sigma^2\right\}$$

The normalization

The 2nd q-moment

$$D = \int \mathrm{d}x\, p(x)^q$$

The probability distribution has its own entropy.

q-EXPECTATION AND q-NORMAL DISTRIBUTION

q-expectation : $E_q[f(x)] = \Omega_q^2 \int dx\, p(x)^q f(x)$

→ 1st q-moment μ, 2nd q-moment σ^2

Tsallis entropy : $S_q = \frac{1}{1-q}\left\{\int dx\, p(x)^q - 1\right\}$

$0 < q < 3$

q-normal distribution

$$p_q(x) = \frac{1}{Z_q}\left\{1 - \frac{1-q}{3-q}\frac{(x-\mu)^2}{\sigma^2}\right\}^{\frac{1}{1-q}}$$ where $Z_q = A_q \sigma$ and

$$\begin{cases} A_q = \dfrac{\sqrt{\frac{3-q}{q-1}\pi}\;\Gamma\left(\frac{1}{q-1}-\frac{1}{2}\right)}{\Gamma\left(\frac{1}{q-1}\right)} & \text{for } 1 \le q < 3 \\ A_q = \dfrac{\sqrt{\frac{3-q}{1-q}\pi}\;\Gamma\left(\frac{2-q}{1-q}\right)}{\Gamma\left(\frac{2-q}{1-q}+\frac{1}{2}\right)} & \text{for } 0 < q < 1 \end{cases}.$$

An escort distribution : $\Omega_q^2\, p_q(x)^q = \frac{2}{3-q}\frac{1}{Z_q}\left\{1 - \frac{1-q}{3-q}\frac{(x-\mu)^2}{\sigma^2}\right\}^{\frac{q}{1-q}}$,

where $\Omega_q^2 = \frac{2}{3-q} Z_q^{q-1}$ for q-normal distribution

- **q-normal distribution**
  - for q = 1, the normal distribution

$$p_1(x)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

  - for q = 2, Cauchy distribution

$$p_2(x)=\frac{1}{B\left(\frac{1}{2},\frac{1}{2}\right)\sigma}\left\{1+\frac{(x-\mu)^2}{\sigma^2}\right\}^{-1}$$

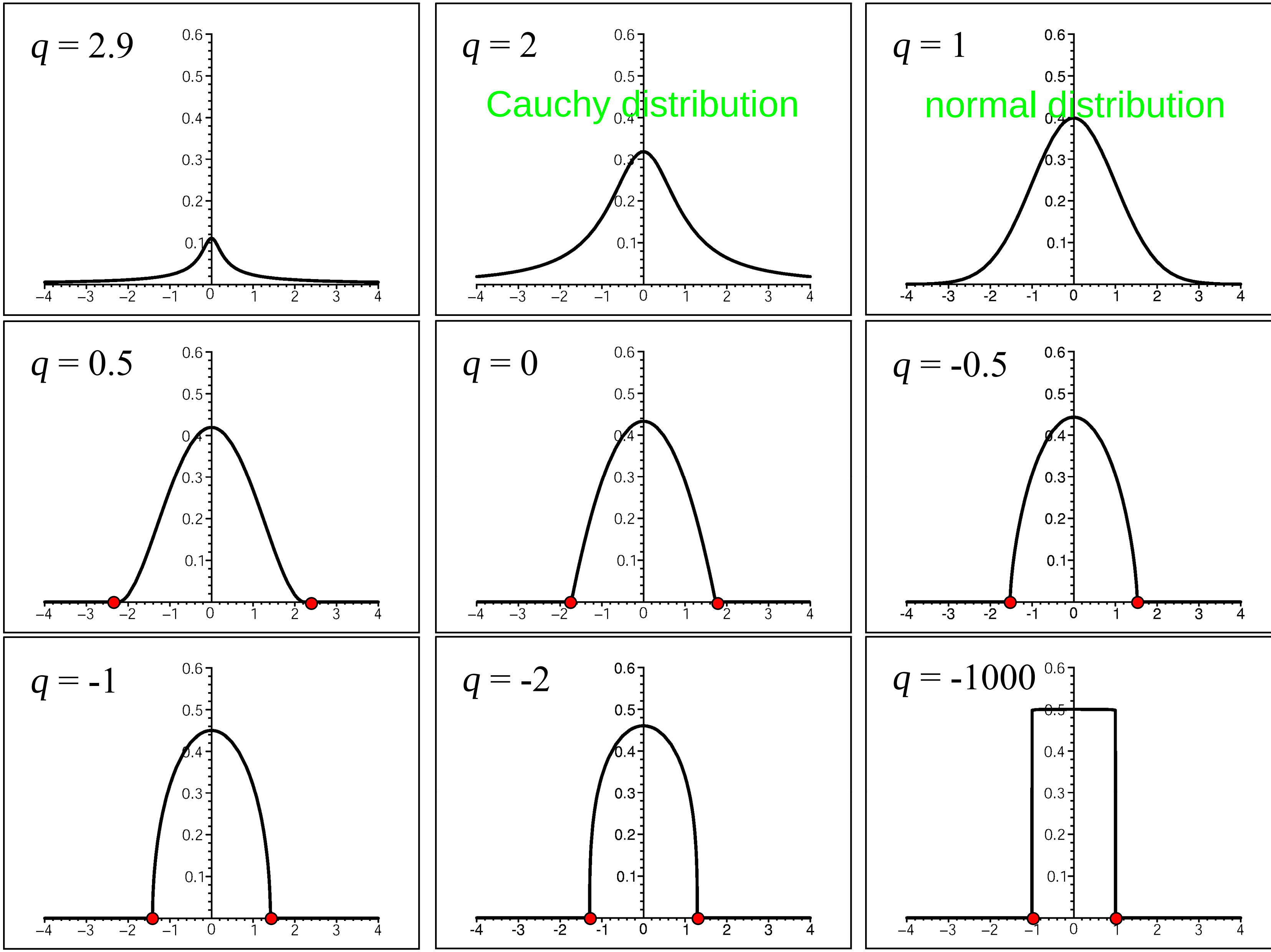
  - for $q = 1+\frac{2}{n+1}$, t-distribution

$$p_{1+\frac{2}{n+1}}(x)=\frac{1}{\sqrt{n}B\left(\frac{n}{2},\frac{1}{2}\right)\sigma}\left\{1+\frac{1}{n}\frac{(x-\mu)^2}{\sigma^2}\right\}^{-\frac{n+1}{2}}$$

If the covariance doesn’t exist, σ should be interpreted as the scale factor.

The q relates the finiteness of the system and the dimension of the probability distribution.

q-normal distribution (The red points show cutoffs.)



NATURAL COORDINATE AND POTENTIAL FUNCTION

The q-normal distribution can be represented as

$$p_q(x) = \{1 + (1-q)(\xi^1 x + \xi^2 x^2 - \psi_q)\}^{\frac{1}{1-q}} .$$

Natural coordinate system : $\begin{cases} \xi^1 = \Omega_q^2 \, \theta^1 \\ \xi^2 = \Omega_q^2 \, \theta^2 \end{cases}$ where $\begin{cases} \theta^1 = \frac{\mu}{\sigma^2} \\ \theta^2 = -\frac{1}{2}\frac{1}{\sigma^2} \end{cases}$.

$$\left[\frac{\partial}{\partial \xi^1} , \frac{\partial}{\partial \xi^2} \right] = 0$$

q-logarithmic likelihood : $L_q p_q = \dfrac{1}{1-q}\left(p_q^{1-q} - 1\right) = \xi^1 x + \xi^2 x^2 - \psi_q(\xi)$,

Potential function : $\psi_q(\xi) = -\dfrac{1}{4}\dfrac{(\xi^1)^2}{\xi^2} - \dfrac{1}{1-q}\left\{\dfrac{3-q}{2}\Omega_q^2 - 1\right\}$

$$\begin{cases} \partial_1 \psi_q(\xi) = -\frac{1}{2}\frac{\xi^1}{\xi^2} = \mathrm{E}_q[x] \\ \partial_2 \psi_q(\xi) = \frac{1}{4}\left(\frac{\xi^1}{\xi^2}\right)^2 - \Omega_q^2 \frac{1}{2\xi^2} = \mathrm{E}_q[x^2] \end{cases}$$

, where $\partial_i = \dfrac{\partial}{\partial \xi^i}$.

METRIC TENSOR

$$\begin{cases} g_{ij} = \partial_i \partial_j \psi_q(\xi) = -\partial_i \partial_j L_q p_q & \text{from the potential function} \\ g_{ij} = \mathrm{E}_q[-\partial_i \partial_j L_q p_q] = q\mathrm{E}_q\left[p_q^{\frac{q-1}{2}} (\partial_i L_q p_q) p_q^{\frac{q-1}{2}} (\partial_j L_q p_q) \right] & \text{from } q\text{-expectation} \end{cases}$$

$$g_{ij} = q\mathrm{E}_q\left[p_q^{\frac{q-1}{2}} (\partial_i L_q p_q) p_q^{\frac{q-1}{2}} (\partial_j L_q p_q) \right]$$

$$= q\Omega_q^2 \int dx\, p_q^{-1} (\partial_i p_q)(\partial_j p_q) = q\Omega_q^2 \mathrm{E}[(\partial_i \log p_q)(\partial_j \log p_q)]$$

Scale factor   Fisher metric

→ Non-Fisher metric !!

In the matrix form,

$$g_{ij} = \begin{pmatrix} -\frac{1}{2}\frac{1}{\xi^2} & \frac{1}{2}\frac{\xi^1}{(\xi^2)^2} \\ \frac{1}{2}\frac{\xi^1}{(\xi^2)^2} & -\frac{1}{2}\frac{(\xi^1)^2}{(\xi^2)^3} + \frac{1}{3-q}\Omega_q^2 \frac{1}{(\xi^2)^2} \end{pmatrix}.$$

The inverse of metric tensor :

$$g^{ij} = (3-q)\Omega_q^{-2} \begin{pmatrix} (\xi^1)^2 - \frac{2}{3-q}\Omega_q^2 \xi^2 & \xi^1 \xi^2 \\ \xi^1 \xi^2 & (\xi^2)^2 \end{pmatrix}.$$

METRIC TENSOR AND DIVERGENCE

For the exponential family : $p(x|\xi) = \exp\{\xi^i X_i - \psi(\xi)\}$, the expectation : $\eta_i = \mathrm{E}[X_i]$

$$
\begin{aligned}
D(p(x|\xi)\| \, p(x|\xi+\rho)) &= \int \mathrm{d}x \; p(x|\xi) \log \frac{p(x|\xi)}{p(x|\xi+\rho)} \\
&\simeq \frac{1}{2} \frac{\partial^2}{\partial\rho^i \partial\rho^j} D(p(x|\xi)\| \, p(x|\xi+\rho)) \bigg|_{\rho=0} \rho^i \rho^j \\
&= \frac{1}{2} \left\{ \int \mathrm{d}x \; p(x|\xi) \frac{\partial \log p(x|\xi+\rho)}{\partial\rho^i} \frac{\partial \log p(x|\xi+\rho)}{\partial\rho^j} \bigg|_{\rho=0} \right\} \rho^i \rho^j \\
&= \frac{1}{2} \mathrm{E}\left[\frac{\partial \log p(x|\xi+\rho)}{\partial\rho^i} \frac{\partial \log p(x|\xi+\rho)}{\partial\rho^j} \bigg|_{\rho=0} \right] \rho^i \rho^j = \frac{1}{2} g_{ij} \rho^i \rho^j
\end{aligned}
$$

$$
\begin{aligned}
D(p(x|\xi)\| \, p(x|\xi+\rho)) &= \psi(\xi+\rho) - \psi(\xi) - \rho^i \eta_i \\
&\simeq \left(\frac{\partial \psi(\xi+\rho)}{\partial\rho^i} \bigg|_{\rho=0} - \eta_i \right) \rho^i + \frac{1}{2} \frac{\partial^2 \psi(\xi+\rho)}{\partial\rho^i \partial\rho^j} \bigg|_{\rho=0} \rho^i \rho^j \\
&= \frac{1}{2} \frac{\partial^2 \psi(\xi+\rho)}{\partial\rho^i \partial\rho^j} \bigg|_{\rho=0} \rho^i \rho^j
\end{aligned}
$$

⇒ g_{ij} : positive definite and symmetric !!

However, for the case of q-normal distributions, …

METRIC TENSOR AND q-DIVERGENCE

$$\frac{\partial L_q p_q(x|\xi)}{\partial \xi^i} = p(x|\xi)^{-q}\frac{\partial p_q(x|\xi)}{\partial \xi^i}$$

, the q-expectation : $\eta_i = \mathrm{E}_q[X_i]$

$$\begin{aligned}
D_q(p_q(x|\xi)\| \, p_q(x|\xi+\rho)) &= \frac{\Omega_q^2(\xi)}{1-q}\left\{1 - \int \mathrm{d}x \; p_q(x|\xi)^q p_q(x|\xi+\rho)^{1-q}\right\} \\
&\simeq \frac{1}{2}\frac{\partial^2}{\partial\rho^i\partial\rho^j} D_q(p_q(x|\xi)\| \, p_q(x|\xi+\rho))\Big|_{\rho=0}\rho^i\rho^j \\
&= \frac{1}{2}\left\{q\Omega_q^2(\xi)\int \mathrm{d}x \; \frac{1}{p_q(x|\xi)}\frac{\partial p_q(x|\xi+\rho)}{\partial\rho^i}\frac{\partial p_q(x|\xi+\rho)}{\partial\rho^j}\Big|_{\rho=0}\right\}\rho^i\rho^j \\
&= \frac{1}{2}\mathrm{E}_q\left[p_q(x|\xi)^{\frac{q-1}{2}}\frac{\partial L_q p_q(x|\xi+\rho)}{\partial\rho^i} p_q(x|\xi)^{\frac{q-1}{2}}\frac{\partial L_q p_q(x|\xi+\rho)}{\partial\rho^j}\Big|_{\rho=0}\right]\rho^i\rho^j
\end{aligned}$$

$$\begin{aligned}
D_q(p_q(x|\xi)\| \, p_q(x|\xi+\rho)) &= \psi_q(\xi+\rho) - \psi_q(\xi) - \rho^i\eta_i \\
&\simeq \left(\frac{\partial\psi_q(\xi+\rho)}{\partial\rho^i}\Big|_{\rho=0} - \eta_i\right)\rho^i + \frac{1}{2}\frac{\partial^2\psi_q(\xi+\rho)}{\partial\rho^i\partial\rho^j}\Big|_{\rho=0}\rho^i\rho^j \\
&= \frac{1}{2}\frac{\partial^2\psi_q(\xi+\rho)}{\partial\rho^i\partial\rho^j}\Big|_{\rho=0}\rho^i\rho^j
\end{aligned}$$

➡ g_{ij} : positive definite and symmetric but non-Fisher metric !!

DIVERGENCE AND *q*-PYTHAGOREAN THEOREM

q-divergence :

$$D_q(p(x|\xi^{(A)})\| p(x|\xi^{(B)})) = \frac{\Omega_q^{(A)2}}{1-q}\left\{1-\int dx\, p(x|\xi^{(A)})^q p(x|\xi^{(B)})^{1-q}\right\}$$

$$= q\Omega_q^{(A)2}\, D_{\alpha=1-2q}(p(x|\xi^{(A)})\| p(x|\xi^{(B)}))$$

$$= \varphi_q^{(A)} + \psi_q^{(B)} - \eta_i^{(A)}\xi^{(B)i}$$

Legendre transformation : $\varphi_q = \eta_i \xi^i - \psi_q$ and $\eta_i = \frac{\partial \psi_q(\xi)}{\partial \xi^i}$

α-divergence :

$$D_\alpha(p(x|\xi^{(A)})\| p(x|\xi^{(B)})) = \frac{4}{1-\alpha^2}\left\{1-\int dx\, p(x|\xi^{(A)})^{\frac{1-\alpha}{2}} p(x|\xi^{(B)})^{\frac{1+\alpha}{2}}\right\}$$

q-Pythagorean theorem :

$$D_q(p(x|\xi^{(A)})\| p(x|\xi^{(B)})) + D_q(p(x|\xi^{(B)})\| p(x|\xi^{(C)})) - D_q(p(x|\xi^{(A)})\| p(x|\xi^{(C)}))$$

$$= (\xi^{(C)i} - \xi^{(B)i})\left(\eta_i^{(A)} - \eta_i^{(B)}\right)$$

PROBABILITY AMPLITUDE

Probability amplitude : $u_q = 2\sqrt{p_q}$.

The 1st order derivative with respect to the natural coordinate :

$$u_{q,i} = \partial_i\left(2\sqrt{p_q}\right) = p_q^{-\frac{1}{2}}(\partial_i p_q) = p_q^{\frac{q}{2}}\left\{p_q^{\frac{q-1}{2}}(\partial_i L_q p_q)\right\} ,$$

Identity : $\Omega_q^2 \int \mathrm{d}x \; u_q u_{q,i} = 2\{\mathrm{E}_q[x^i] - \partial_i \psi_q(\xi)\} = 0$.

The 2nd order derivative with respect to the natural coordinates :

$$u_{q,ij} = p_q^{-\frac{1}{2}}\left\{\left(q - \frac{1}{2}\right)p_q^{2q-1}(\partial_i L_q p_q)(\partial_j L_q p_q) + p_q^q(\partial_i \partial_j L_q p_q)\right\} .$$

Metric tensor : $g_{ij} = q\Omega_q^2 \int \mathrm{d}x \; u_{q,i} u_{q,j} = -q\Omega_q^2 \int \mathrm{d}x \; u_q u_{q,ij}$.

The 1st kind Christoffel symbol :

$$\Gamma_{kij} = \frac{1}{2}\{\partial_i g_{kj} + \partial_j g_{ik} - \partial_k g_{ij}\} = \frac{1}{2}\partial_i \partial_j \partial_k \psi_q(\xi)$$

$$= \left(q - \frac{1}{2}\right) q\Omega_q^2 \int \mathrm{d}x \; p_q^{-\frac{1}{2}} u_{q,i} u_{q,j} u_{q,k}$$

$$+ \frac{1}{2}\{\Omega_q^{-2}\partial_k \Omega_q^2 g_{ij} + \Omega_q^{-2}\partial_j \Omega_q^2 g_{ik} + \Omega_q^{-2}\partial_i \Omega_q^2 g_{kj}\} .$$

GAUGE THEORETIC DERIVATION OF α-CONNECTION

Local gauge transformation : $u_{q,i}^{(\alpha)} = e^{-\frac{\alpha}{2}L_q p_q} u_{q,i}$.

α-conjugation : $\alpha^{\#} = -\alpha$, $(-\alpha)^{\#} = \alpha$.

For example, $\left(u_{q,i}^{(\alpha)}\right)^{\#} = u_{q,i}^{(-\alpha)} = e^{\frac{\alpha}{2}L_q p_q} u_{q,i}$.

The local gauge transformation becomes U(1) under α-conjugation.

Metric tensor redefined : $g_{ij} = q\Omega_q^2 \int \mathrm{d}x \left(u_{q,i}^{(\alpha)}\right)^{\#} u_{q,j}^{(\alpha)} = q\Omega_q^2 \int \mathrm{d}x \, u_{q,i} u_{q,j}$.

Covariant derivative before local gauge transformation :

$$D_j u_{q,i} = \partial_j u_{q,i} - \Gamma_{ij}^{\ell} u_{q,\ell} .$$

Covariant derivative after local gauge transformation :

$$\begin{aligned} D_j^{(\alpha)} u_{q,i}^{(\alpha)} &= \partial_j u_{q,i}^{(\alpha)} - \Gamma_{ij}^{(\alpha)\ell} u_{q,\ell}^{(\alpha)} \\ &= e^{-\frac{\alpha}{2}L_q p_q}\left(u_{q,ij} - \Gamma_{ij}^{(\alpha)\ell} u_{q,\ell} - \frac{\alpha}{2} p_q^{-q+\frac{1}{2}} u_{q,j} u_{q,i} \right) . \end{aligned}$$

$$\left\{ \begin{array}{l} \text{Assumption : } \Gamma_{ij}^{(\alpha)\ell} = \Gamma_{ij}^{\ell} - \frac{\alpha}{2} A_{ij}^{\ell} \quad \text{under the local gauge transformation} \\ \qquad\qquad A_{ij}^{\ell} \text{ is called a gauge field.} \\ \text{Requirement : } D_j^{(\alpha)} u_{q,i}^{(\alpha)} = e^{-\frac{\alpha}{2} L_q p_q} D_j u_{q,i} \ . \end{array} \right.$$

← how to determine A_{ij}^{ℓ}

From the assumption, $D_j^{(\alpha)} u_{q,i}^{(\alpha)} = e^{-\frac{\alpha}{2} L_q p_q} \left\{ D_j u_{q,i} + \frac{\alpha}{2} \left(A_{ij}^{\ell} u_{q,\ell} - p_q^{-q+\frac{1}{2}} u_{q,j} u_{q,i} \right) \right\}$.

Then $$q\Omega_q^2 \int \mathrm{d}x \left(u_{q,k}^{(\alpha)} \right)^{\#} D_j^{(\alpha)} u_{q,i}^{(\alpha)}$$

$$= q\Omega_q^2 \int \mathrm{d}x \; u_{q,k} D_j u_{q,i} + \frac{\alpha}{2} \left(A_{ij}^{\ell} g_{\ell k} - q\Omega_q^2 \int \mathrm{d}x \; p_q^{-q+\frac{1}{2}} u_{q,k} u_{q,j} u_{q,i} \right) .$$

From the requirement, $q\Omega_q^2 \int \mathrm{d}x \left(u_{q,k}^{(\alpha)} \right)^{\#} D_j^{(\alpha)} u_{q,i}^{(\alpha)} = q\Omega_q^2 \int \mathrm{d}x \; u_{q,k} D_j u_{q,i}$.

Therefore we have $A_{kij} = q\Omega_q^2 \int \mathrm{d}x \; p_q^{-q+\frac{1}{2}} u_{q,k} u_{q,i} u_{q,j}$, where $A_{kij} = A_{ij}^{\ell} g_{\ell k}$.

Dualistic structure : $\Gamma^{(\alpha)}_{kij} + \Gamma^{(-\alpha)}_{kij} = 2\Gamma_{kij} = \partial_k g_{ij}$.

Covariant derivative of metric tensor with α-connection :

$$D^{(\alpha)}_k g_{ij} = D^{(\alpha)}_k \left\{ q\Omega_q^2 \int \mathrm{d}x \left(u^{(\alpha)}_{q,i} \right)^{\#} u^{(\alpha)}_{q,j} \right\} = \alpha A_{ijk} .$$

Derived α-connection :

$$\Gamma^{(\alpha)}_{111} = 0$$

$$\Gamma^{(\alpha)}_{112} = (1 - \alpha\Omega_q^2)\frac{1}{4}\frac{1}{(\xi^2)^2}$$

$$\Gamma^{(\alpha)}_{122} = -(1 - \alpha\Omega_q^2)\frac{1}{2}\frac{\xi^1}{(\xi^2)^3}$$

$$\Gamma^{(\alpha)}_{222} = (1 - \alpha\Omega_q^2)\frac{3}{4}\frac{(\xi^1)^2}{(\xi^2)^4} - \left(1 - \alpha\Omega_q^2\frac{3-q}{5-3q}\right)\Omega_q^2\frac{5-q}{2(3-q)^2}\frac{1}{(\xi^2)^3} ,$$

α-RIEMANN TENSOR

Definition : $\left[D_k^{(\alpha)} , D_j^{(\alpha)} \right] U_i = R_{ijk}^{(\alpha)\ell} U_\ell = g^{\ell m} R_{mijk}^{(\alpha)} U_\ell$,

α-Riemann tensor (the others are 0.) :

$$R_{1212}^{(\alpha)} = \frac{1}{4}\frac{1}{3-q}\frac{1}{(\xi^2)^3} + \frac{\alpha}{2}\Omega_q^2 \frac{(1-q)(5-2q)}{(3-q)(5-3q)}\frac{1}{(\xi^2)^3} - \frac{\alpha^2}{4}\Omega_q^2 \frac{q\Omega_q^2}{5-3q}\frac{1}{(\xi^2)^3}$$

$$R_{2112}^{(\alpha)} = -\frac{1}{4}\frac{1}{3-q}\frac{1}{(\xi^2)^3} + \frac{\alpha}{2}\Omega_q^2 \frac{(1-q)(5-2q)}{(3-q)(5-3q)}\frac{1}{(\xi^2)^3} + \frac{\alpha^2}{4}\Omega_q^2 \frac{q\Omega_q^2}{5-3q}\frac{1}{(\xi^2)^3}$$

$$R_{1221}^{(\alpha)} = -R_{1212}^{(\alpha)}$$

$$R_{2121}^{(\alpha)} = -R_{2112}^{(\alpha)}$$

$$R_{2212}^{(\alpha)} = -\alpha\Omega_q^2 \frac{(1-q)(5-2q)}{(3-q)(5-3q)}\frac{\xi^1}{(\xi^2)^4}$$

$$R_{2221}^{(\alpha)} = \alpha\Omega_q^2 \frac{(1-q)(5-2q)}{(3-q)(5-3q)}\frac{\xi^1}{(\xi^2)^4} ,$$

$$R_{1212}^{(\alpha)} \neq -R_{2112}^{(\alpha)}$$

$$R_{2212}^{(\alpha)} \neq 0$$

$$R_{2221}^{(\alpha)} \neq 0 .$$

The α-Riemann tensor isn't antisymmetric, that is, the q-normal distribution is **not conjugate symmetric**.

α-RICCI TENSOR

Definition : $R^{(\alpha)}_{ij} = g^{k\ell} R^{(\alpha)}_{\ell ikj} = R^{(\alpha)k}{}_{ikj}$.

α -Ricci tensor :

$$R^{(\alpha)}_{11} = \Omega_q^{-2} \frac{1}{4} \frac{1}{\xi^2} + \frac{\alpha}{4} \frac{(q-1)(5-q)}{5-3q} \frac{1}{\xi^2} - \frac{\alpha^2}{4} \Omega_q^2 \frac{q(3-q)}{5-3q} \frac{1}{\xi^2}$$

$$R^{(\alpha)}_{12} = -\Omega_q^{-2} \frac{1}{4} \frac{\xi^1}{(\xi^2)^2} - \frac{\alpha}{4} \frac{(q-1)(5-q)}{5-3q} \frac{\xi^1}{(\xi^2)^2} + \frac{\alpha^2}{4} \Omega_q^2 \frac{q(3-q)}{5-3q} \frac{\xi^1}{(\xi^2)^2}$$

$$R^{(\alpha)}_{21} = -\Omega_q^{-2} \frac{1}{4} \frac{\xi^1}{(\xi^2)^2} - \frac{\alpha}{4} \frac{(q-1)(5-q)}{5-3q} \frac{\xi^1}{(\xi^2)^2} + \frac{\alpha^2}{4} \Omega_q^2 \frac{q(3-q)}{5-3q} \frac{\xi^1}{(\xi^2)^2}$$

$$\begin{aligned} R^{(\alpha)}_{22} = {} & \Omega_q^{-2} \frac{1}{4} \left\{ \frac{(\xi^1)^2}{(\xi^2)^3} - \Omega_q^2 \frac{2}{3-q} \frac{1}{(\xi^2)^2} \right\} \\ & + \frac{\alpha}{4} \frac{(q-1)(5-q)}{5-3q} \left\{ \frac{(\xi^1)^2}{(\xi^2)^3} + \Omega_q^2 \frac{2}{3-q} \frac{1}{(\xi^2)^2} \right\} \\ & - \frac{\alpha^2}{4} \Omega_q^2 \frac{q(3-q)}{5-3q} \left\{ \frac{(\xi^1)^2}{(\xi^2)^3} - \Omega_q^2 \frac{2}{3-q} \frac{1}{(\xi^2)^2} \right\} . \end{aligned}$$

$$R^{(\alpha)}_{12} = R^{(\alpha)}_{21}$$

The α-Ricci tensor is symmetric, that is,
the q-normal distribution is **statistically equiaffine**.

The Jeffreys' prior, which plays an important role in statistical inference, is generalized to the α-parallel prior by J. Takeuchi and S. Amari (2005) from the information geometrical point of view.

However the α-parallel prior doesn't always exist for any value of α. So they give the sufficient condition for the existence of the α-parallel prior such that, if a statistical model is conjugate symmetric, the model is staistically equiaffine.

Then they leave the open problem such that whether there exists a statistically equiaffine natural model which is not conjugate symmetric or not.

⇒ There exists such a model and the corresponding α-parallel prior is,

$$\exp\left\{\alpha\frac{(5-2q)(3-q)}{5-3q}\Omega_q^2 S_q\right\}\sqrt{g}\,(\mathrm{d}\xi^1\wedge\mathrm{d}\xi^2)$$

$$D_i^{(\alpha)}\left[\exp\left\{\alpha\frac{(5-2q)(3-q)}{5-3q}\Omega_q^2 S_q\right\}\sqrt{g}\,(\mathrm{d}\xi^1\wedge\mathrm{d}\xi^2)\right]=0$$

THE JEFFREYS' PRIOR

In Bayesian statistical inference

Bayes rule for the posterior probability : $P(A|E) = \frac{P(A)}{P(E)} \int \mathrm{d}^d\Theta \; \omega(\Theta) P(E|\Theta)$

A prior density on the parameter space : $\omega(\Theta)$, $A \in$ the model family

A collection of outcomes (i.i.d.) : E

Ignorance on the prior density : $\omega(\Theta)$

→ The uniform distribution on the parameter space

→ However, if a prior is uniform in the parameter space, the probability of a model family will depend on the arbitrary parametrization.

→ The uniform distribution NOT on parameters BUT distributions

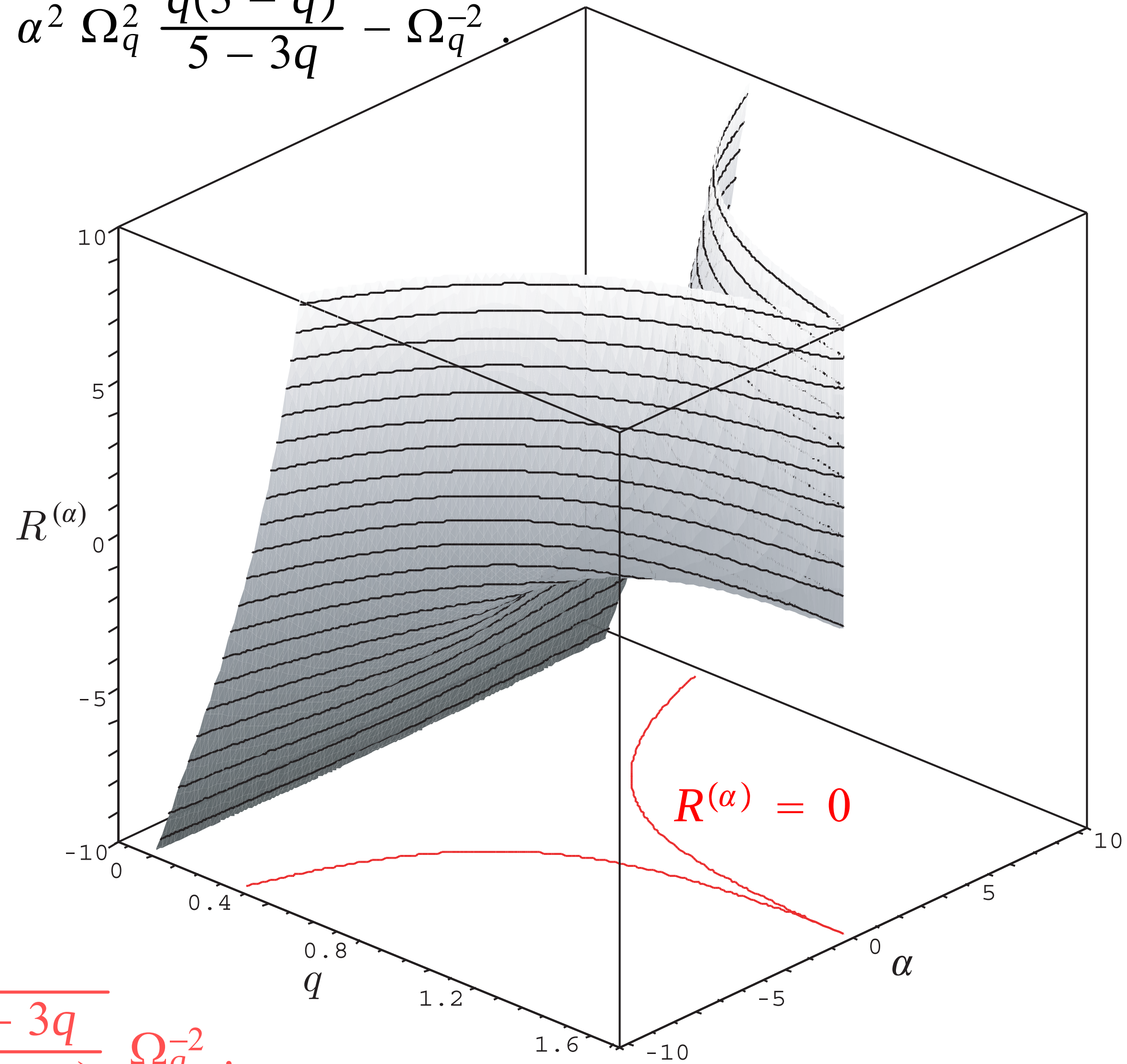
→ The invariant measure on the parameter space under reparametrizations

→ The Jeffreys' prior : $\sqrt{g}\, \mathrm{d}^d\Theta$

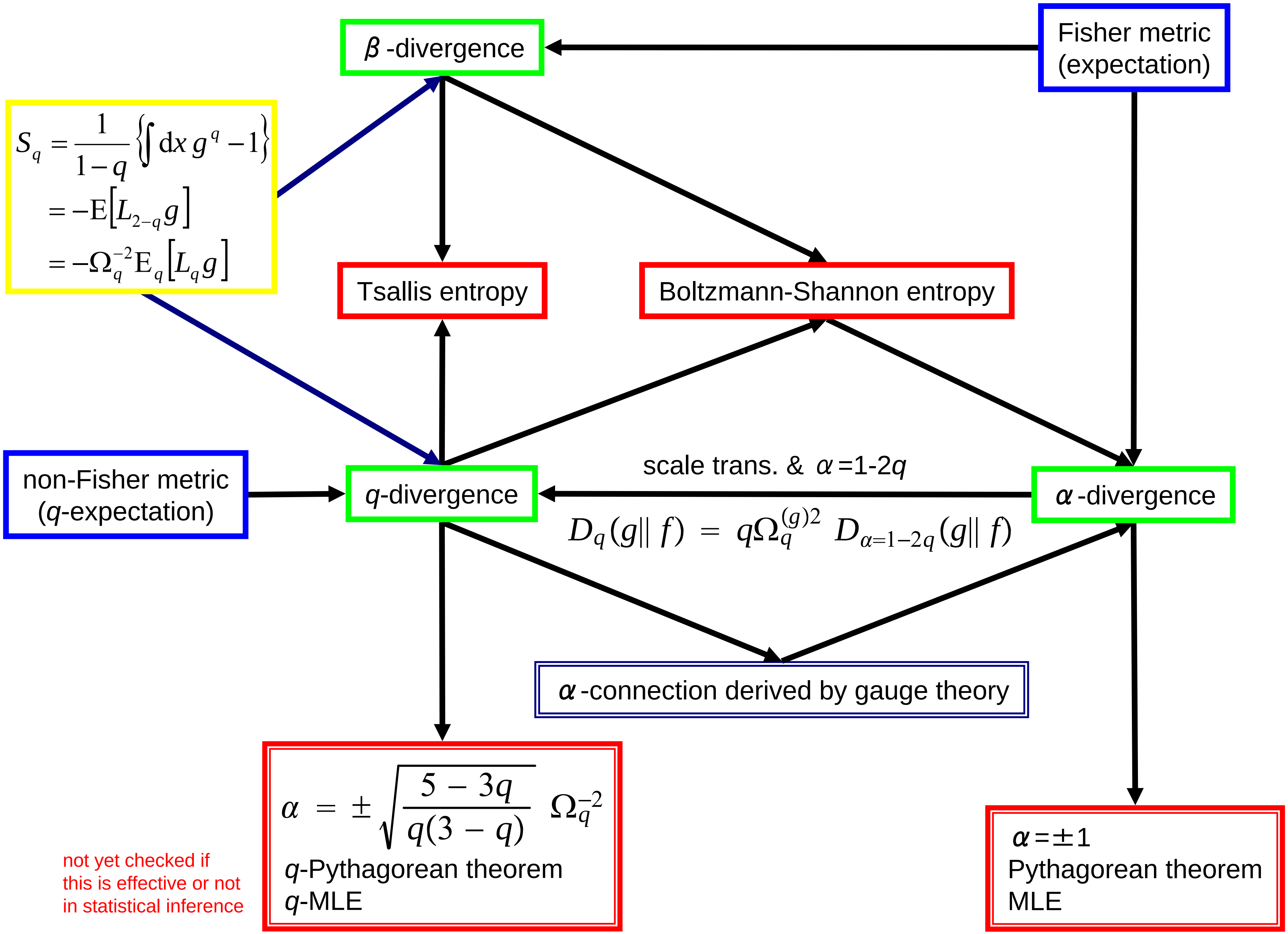
α-RICCI SCALAR CURVATURE

α-Ricci scalar curvature :

$$R^{(\alpha)} = g^{ij} R_{ji}^{(\alpha)} = R_{\ i}^{(\alpha)i} = \alpha^2 \, \Omega_q^2 \, \frac{q(3-q)}{5-3q} - \Omega_q^{-2} \, .$$



$R^{(\alpha)} = 0$ at $\alpha = \pm \sqrt{\dfrac{5-3q}{q(3-q)}} \, \Omega_q^{-2}$.


β-divergence
Fisher metric (expectation)
$S_q = \frac{1}{1-q}\left\{\int dx\, g^q - 1\right\} = -\mathrm{E}\left[L_{2-q} g\right] = -\Omega_q^{-2}\mathrm{E}_q\left[L_q g\right]$
Tsallis entropy
Boltzmann-Shannon entropy
non-Fisher metric (q-expectation)
q-divergence
scale trans. & α=1-2q
α-divergence
$D_q(g\|f) = q\Omega_q^{(g)2} D_{\alpha=1-2q}(g\|f)$
α-connection derived by gauge theory
$\alpha = \pm\sqrt{\frac{5-3q}{q(3-q)}}\ \Omega_q^{-2}$
q-Pythagorean theorem
q-MLE
not yet checked if this is effective or not in statistical inference
$\alpha = \pm 1$
Pythagorean theorem
MLE