On the Geometry of Riemann-Finsler surfaces

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1. Minkowski planes

A **Minkowski plane** is the vector space R^2 endowed with a Minkowski norm.

A **Minkowski norm** on R^2 is a nonnegative real valued function $F : \mathbf{R}^2 \to [0, \infty)$ with the properties

1. F is C^∞ on $\widetilde{\mathbf{R}^2} = \mathbf{R}^2 \setminus \{0\}$,
2. **1-positive homogeneity** : $F(\lambda y) = \lambda F(y), \ \forall \lambda > 0, \ y \in \mathbf{R}^2$,
3. **strong convexity:** the Hessian matrix $g_{ij}(y) = \frac{1}{2} \frac{\partial^2 F^2(y)}{\partial y^i \partial y^j}$ is positive definite on $\widetilde{\mathbf{R}^2}$.

The **indicatrix** $S := \{y \in \mathbf{R}^2 : F(y) = 1\}$ is a closed, strictly convex, smooth curve around the origin $y = 0$.

Let (M, F) be a Minkowski plane.

Cartan tensor

(1) $$A_{ijk}(y) := \frac{F(y)}{4} \frac{\partial^3 F^2(y)}{\partial y^i \partial y^j \partial y^k}, \quad i, j, k \in \{1, 2\}.$$

The Minkowski norm F on $\mathbf{R}^2$ induces a Riemannian metric $\hat{g}$ on the punctured plane $\widetilde{R}^2$ by

(2) $$\hat{g} := g_{ij}(y) dy^i \otimes dy^j.$$

The **Cartan scalar (main scalar)** $I : \widetilde{R}^2 \to \mathbf{R}$ is defined by

(3) $$I(y) = A_{ijk}(y) \frac{dy^i}{dt} \frac{dy^j}{dt} \frac{dy^k}{dt}.$$

The volume form of the Riemannian metric $\hat{g}$:

(4) $$dV = \sqrt{g} dy^1 \wedge dy^2,$$

where $\sqrt{g} = \sqrt{det(g_{ij})}$.
The induced Riemannian volume form on the indicatrix submanifold S is

(5) $$ds = \sqrt{g}(y^1 \dot{y}^2 - y^2 \dot{y}^1) dt.$$

Along S the 1-form ds coincides with

(6) $$d\theta = \frac{\sqrt{g}}{F^2}(y^1 dy^2 - y^2 dy^1).$$

The parameter θ is called the **Landsberg angle.**

2. Riemannian Length of the Indicatrix

$$L := \int_S ds = \int_{\mathbf{S}^1} \frac{\sqrt{g}}{F^2}(y^1 dy^2 - y^2 dy^1).$$

Remark that

(7) $$\frac{\sqrt{g}}{F^2}(y^1 \dot{y}^2 - y^2 \dot{y}^1) = \sqrt{g_{ij}(y)\dot{y}^i \dot{y}^j},$$

i.e. measure the Riemannian arc length of the indicatrix, regarded as a curve in $\widetilde{\mathbf{R}^2}$, by the Riemannian metric $\hat{g}$. L is typically NOT equal to 2π as in the case of Riemannian surfaces. This fact was remarked for the first time by M. Matsumoto in 1986.

Example 1

Consider a Randers- Minkowski norm

(8) $$F(y^1, y^2) = \sqrt{(y^1)^2 + (y^2)^2} + By^1$$

on $\mathbf{R}^2$, where $B \in [0, 1)$ is a constant parameter.
Polar equation of the indicatrix

(9) $$r = \frac{1}{1 + B\cos\varphi},$$

The indicatrix length is given by the elliptic integral

$$L = \frac{4}{\sqrt{1+B}} \int_0^{\frac{\pi}{2}} \frac{d\mu}{\sqrt{1-k^2 \sin^2 \mu}}, \tag{10}$$

where $\varphi = 2\mu$, and $k := \sqrt{\frac{2B}{1+B}}$.

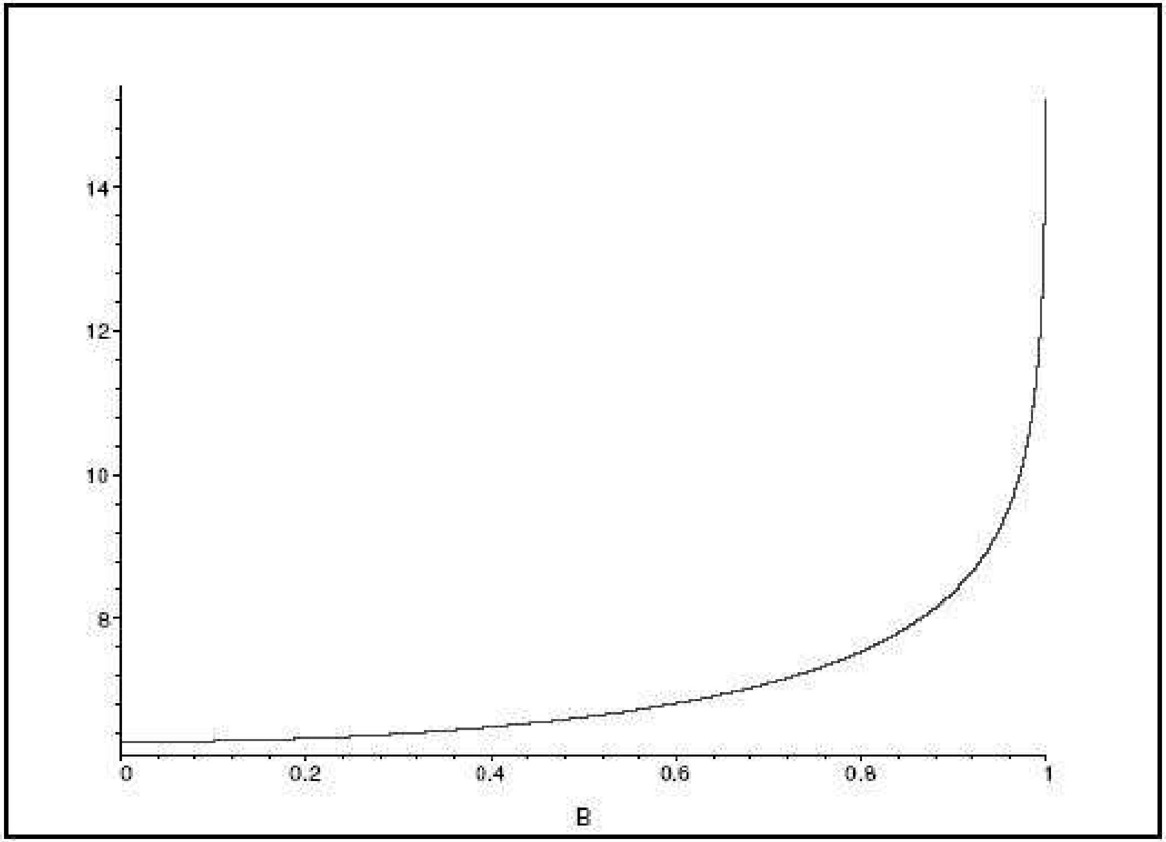


Figure 1. The variation of Riemannian length of the indicatrix for the metric given in Example 1.

Consider the Minkowski norm

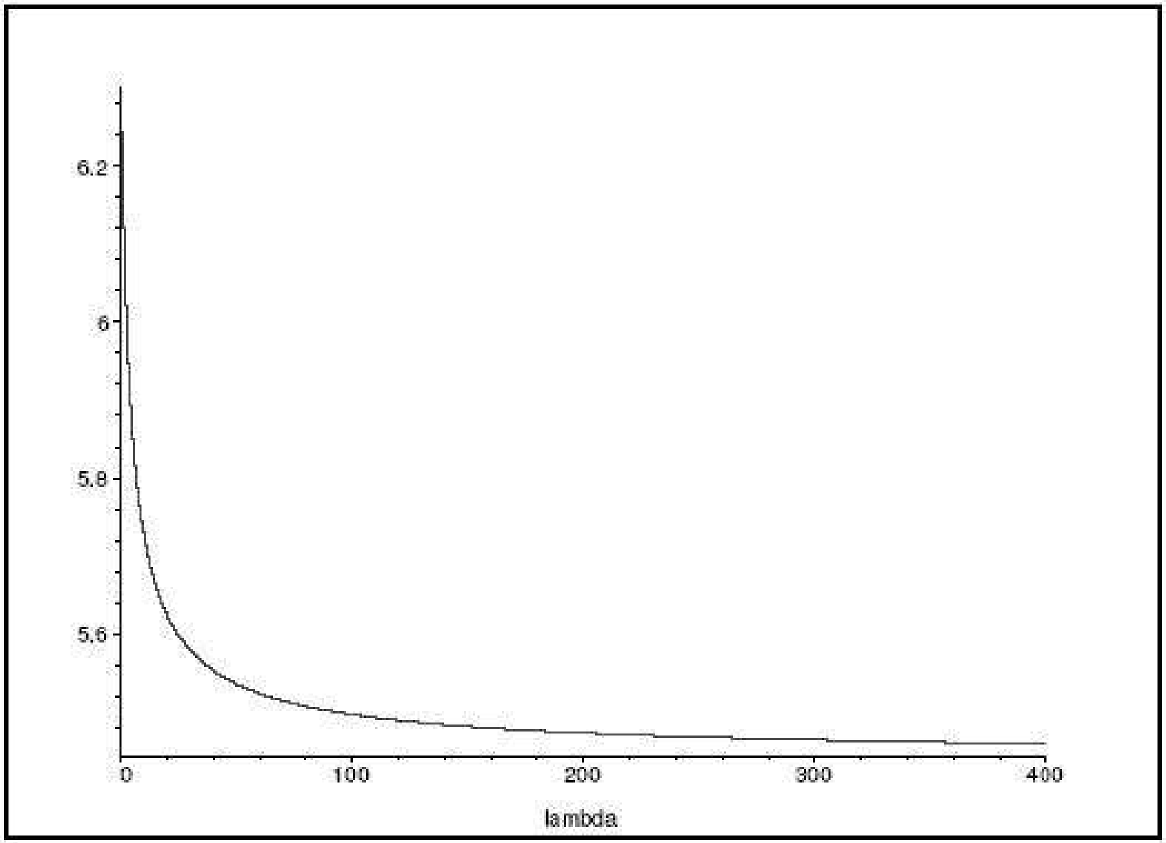
(11) $$F(y) = \sqrt{(y^1)^2 + (y^2)^2 + \lambda\sqrt{(y^1)^4 + (y^2)^4}}, \quad \lambda \geq 0$$

in $\mathbf{R}^2$.

With the substitution $u := \dfrac{y^2}{y^1}$ one obtains the indicatrix length

(12) $$L = 8\int_0^1 \frac{\sqrt{1 + \lambda\frac{(1+u^2)^3}{(1+u^4)^{3/2}} + \lambda^2\frac{3u^2}{1+u^4}}}{1 + u^2 + \lambda\sqrt{1+u^4}}du.$$

$$\lim_{\lambda\to\infty} L = \sqrt{3}\pi.$$



3. Finsler surfaces

A **Finsler surface** is the pair (M, F) where $F : TM \to [0, \infty)$ is $\mathbf{C}^\infty$ on $\widetilde{TM} := TM \backslash \{0\}$ and whose restriction to each tangent plane $T_x M$ is a Minkowski norm.

For each $x \in M$ the quadratic form $ds^2 := g_{ij}(x, y) dy^i \otimes dy^j$ gives a Riemannian metric on the punctured tangent space $\widetilde{T_x M}$. Using the Finslerian fundamental function F we define the *indicatrix bundle* (or *unit sphere bundle*) $SM := \cup_{x \in M} S_x M$, where $S_x M := \{y \in T_x M : F(x, y) = 1\}$. Topologically, $I_x M$ is diffeomorphic with the Euclidean unit sphere S^2 in $\mathbf{R}^3$. Moreover, the above ds^2 induces a Riemannian metric h_x on each $S_x M$.

Since the Finslerian fundamental tensor $g_{ij}(x, y)$ is invariant under the rescaling $y \mapsto \lambda y$, $\lambda > 0$, the inner products in the fibers T_xM are actually identical. This redundacy is removed by working with the pull-back bundle π^*TM over SM.

Riemann–Finsler geometry is the geometry of the pullback bundle (π^*TM, π, SM) endowed with an inner product in each fiber.

The Riemannian case. On a Riemannian manifold (M, g) the metric

$$g = g_{ij}(x) dx^i \otimes dx^j \tag{13}$$

is a specific inner product in each tangent space $T_x M$ viewed as a vector space. Moreover,

$$\hat{g} = g_{ij}(x) dy^i \otimes dy^j \tag{14}$$

is an isotropic Riemannian metric on $T_x M$ viewed as differentiable manifold.

The Riemann-Finslerian case. On a Riemann-Finsler manifold (M, F) the metric

(15) $$g = g_{ij}(x,y)dx^i \otimes dx^j$$

is a family of inner products in each tangent space T_xM viewed as a vector space, parametrized by rays ty, ($t > 0$) which emanate from origin. This is actually a Riemannian metric on π^*TM. Moreover,

(16) $$\hat{g} = g_{ij}(x,y)dy^i \otimes dy^j$$

is a non-isotropic Riemannian metric on T_xM viewed as differentiable manifold, which is invariant along each ray and possibly singular at origin.

4. Moving frame methods

Using the global section $l := \frac{y^i}{F(y)} \frac{\partial}{\partial x^i}$ of $\pi^* TM$, one can construct a positively oriented g-orthonormal frame $\{e_1, e_2\}$ for $\pi^* TM$, called **Berwald frame**, where $g = g_{ij}(x,y) dx^i \otimes dx^j$ is the induced Riemannian metric on the fibers of $\pi^* TM$.

$$e_1 := \frac{1}{\sqrt{g}} \Big(\frac{\partial F}{\partial y^2} \frac{\partial}{\partial x^1} - \frac{\partial F}{\partial y^1} \frac{\partial}{\partial x^2} \Big) = m^1 \frac{\partial}{\partial x^1} + m^2 \frac{\partial}{\partial x^2},$$

$$e_2 := \frac{y^1}{F} \frac{\partial}{\partial x^1} + \frac{y^2}{F} \frac{\partial}{\partial x^2} = l^1 \frac{\partial}{\partial x^1} + l^2 \frac{\partial}{\partial x^2}.$$

5. Chern connection

There exists a torsion free, almost g-compatible connection on the Riemannian manifold (π^*TM, g), called the **the Chern connection of a Finsler surface.**

The Chern connection matrix

(17) $$\begin{pmatrix} {\omega_1}^1 & {\omega_1}^2 \\ {\omega_2}^1 & {\omega_2}^2 \end{pmatrix} = \begin{pmatrix} -I\omega^3 & -\omega^3 \\ \omega^3 & 0 \end{pmatrix}$$

where $I := A_{111} = A(e_1, e_1, e_1)$ is the **Cartan scalar** for Finsler surfaces.

6. Structure equations

The **structure equations** of a Finsler surface

$$\begin{aligned} d\omega^1 &= -I\omega^1 \wedge \omega^3 + \omega^2 \wedge \omega^3 \\ d\omega^2 &= -\omega^1 \wedge \omega^3 \\ d\omega^3 &= K\omega^1 \wedge \omega^2 - J\omega^1 \wedge \omega^3. \end{aligned}$$

The **Bianchi identities**

$$\begin{aligned} J &= I_2 = \frac{1}{F}\Big(y^1 \frac{\delta I}{\delta x^1} + y^2 \frac{\delta I}{\delta x^2}\Big) \\ K_3 &+ KI + J_2 = 0. \end{aligned}$$

7. Parallel translation

Let $\sigma : [a, b] \to M$ be an arbitrary C^∞ piecewise curve.
The (nonlinear) covariant derivative of W along σ is defined by

$$D_T^W W := \left[\frac{dW^i}{dt} + W^j T^k \Gamma^i_{jk}(\sigma(t), W(t))\right] \frac{\partial}{\partial x^i}_{|\sigma(t)},$$

where Γ^i_{jk} are the coefficients of Chern connection. The top letter indicates the reference vector. If it is absent it means that the reference vector is T.
$W(t)$ is said to be **(nonlinearly) parallel** along $\sigma(t)$ if $D_T^W W = 0$.
The **(nonlinear) parallel translation** along $\sigma(t)$ is given by the map

$$P_\sigma : T_{\sigma(a)}M \to T_{\sigma(b)}M, \qquad P_\sigma(v) = w,$$

where $V(t)$ is a nonlinearly parallel vector field along σ with $V(a) = v$, $V(b) = w$.

Remark. For a C^∞ piecewise curve σ on a Finsler manifold (M, F), the parallel translation preserves the Finslerian norm, i.e. if $W(t)$ is parallel along σ, then $F(\sigma(t), W(t))$ is constant.

The Finsler surface (M, F) is called a **Berwald surface** if the parallel translation $P_\sigma : T_{\sigma(a)}M \to T_{\sigma(b)}M$ is a linear isomorphism, where $\sigma : [a, b] \to M$, $\sigma(a) = p \in M$, $\sigma(b) = q \in M$, is a C^∞ piecewise curve.

The Finsler surface (M, F) is called a **Landsberg surface** if the parallel translation

$$P_\sigma : (\widetilde{T_pM}, g_p) \to (\widetilde{T_qM}, g_q) \tag{18}$$

is an isometry, where g_x is the induced Riemannian metric in T_xM, for any $x \in M$.

8. Landsberg surfaces

A Landsberg surface is characterized by $J = 0$, or, equivalently, $I_2 = 0,\ K_3 = -KI$.

Bianchi equations imply

$$\begin{aligned} dI &= I_1\omega^1 + I_3\omega^3 \\ dK &= K_1\omega^1 + K_2\omega^2 - KI\omega^3. \end{aligned}$$

From the general Cartan–Kahler theory of EDS it follows that such structures depend on two functions of two variables (Bryant, 1995).

A Finsler surface is Landsberg if and only if its indicatrix bundle SM is a principal right $SO(2)$-bundle which connection is induced by the Chern connection (Bryant, 1995).

Theorem (Rigidity theorem for Berwald surfaces, Z. Szabo, 1981) Let (M, F) be a connected Berwald surface for which the Finsler structure F is smooth and strongly convex on all $\widetilde{TM}$.

1. If $K \equiv 0$, then F is locally Minkowski everywhere.
2. If $K \not\equiv 0$, then F is Riemannian everywhere.

Theorem (D. Bao 2000) The indicatrix length of a Landsberg surface is constant.

9. Curves on a Finsler surface

Let $\gamma : [a, b] \to M$ be a smooth curve on a Finsler surface (M, F), given by $x^i = x^i(t)$, $T(t) = \dot{\gamma}(t)$, $F(\gamma(t), T(t)) = 1$, for any $t \in [a, b]$.

Prop. For each fixed point $x(t)$, $t \in [a, b]$, on the curve γ, there exists a Finslerian unit length vector field $N(t) \in \widetilde{T_{x(t)}M}$ such that

$$g_{N(t)}(N(t), T(t)) = 0.$$

We define the **tangent geodesic curvature vector over** N by

$$\mathbf{K}^{(N)}(t) := D_T^{(N)} N = (\frac{dN^i}{dt} + \Gamma^i_{jk}(x, N) N^j T^k) \frac{\partial}{\partial x^i}_{|\gamma(t)},$$

the **geodesic curvature over** N of the curve γ by

$$k^{(N)}(t) := [g_N(\mathbf{K}^{(N)}(t), \mathbf{K}^{(N)}(t))]^{\frac{1}{2}},$$

and the **signed geodesic curvature over N** of γ by

$$k_T^{(N)}(t) := -g_N(\mathbf{K}^{(N)}(t), T(t)),$$

where $T(t)$ is considered now as a vector field in the fiber of $\pi^* TM$ over $(\gamma(t), N(t))$.

Prop. If γ is a smooth curve on the Finsler surface (M, F), then the following relations hold good

$$\mathbf{K}^{(N)}(t) = -\frac{1}{\sigma^2(t)}\, k_T^{(N)}(t) T(t),$$
$$k^{(N)}(t) = \frac{1}{\sigma(t)}\, |k_T^{(N)}(t)|,$$

where $\sigma^2(t) = g_N(T, T)$, and $\mathbf{K}^{(N)}(t)$, $k^{(N)}(t)$ are the tangent geodesic curvature vector, and geodesic curvature of γ, respectively.

10. Gauss-Bonnet Theorem

For an arbitrary vector field $V : M \to TM$, $x \mapsto V(x) \in T_xM$, we denote its zeros by $x_\alpha \in M$, $(\alpha = 1, 2, \ldots, k)$, and by i_α the index of V at x_α.

By removing from M the interiors of the geodesic circles S_α^ε (centred at x_α and of radius ε), one obtains the manifold with boundary M_ε.

Remark that in this case, the boundary of M_ε consists of the boundaries of the geodesic circles S_α^ε and the boundary of M.

Assuming that V has all zeros in $M \setminus \partial M$, it follows that V has no zeros on M_ε, and therefore we can normalize it obtainig the following mapping:

$$X = \frac{V}{F(V)} : M_\varepsilon \to SM, \quad x \mapsto \frac{V(x)}{F(V(x))}.$$

Using this X, we can lift M_ε to SM obtainig in this way the 2-dimensional submanifold $X(M_\varepsilon)$ of SM.

Prop. Let (M, F) be a compact oriented Landsberg surface with the smooth boundary ∂M. Let

$$N : \partial M \to SM$$

be the inward pointing Finslerian unit normal on ∂M. Then, we have

$$\int_M K\sqrt{g}dx^1 \wedge dx^2 + \int_{N(\partial M)} {\omega_1}^2 = L \cdot \mathcal{X}(M),$$

where L is the Riemannian length of the indicatrix of (M, F), K and g are the Gauss curvature and the determinant of the fundamental tensor g_{ij} of F, respectively, ${\omega_1}^2$ the Chern connection form, and $\mathcal{X}(M)$ the Euler characteristic of M.

Prop. On the Finsler manifold (M, F) with smooth boundary $\partial M = \gamma : [a, b] \to M$, we have

$$\int_{N(\partial M)} {\omega_1}^2 = \int_\gamma \frac{k_N^{(N)}(t)}{\sigma(t)} dt,$$

where N is the inward pointing normal on γ, ${\omega_1}^2$ the Chern connection form and $\sigma^2 = g_N(T, T)$.

Theorem (Gauss-Bonnet Theorem for Landsberg surfaces)
Let (M, F) be a compact, connected Landsberg surface with unit velocity smooth boundary. Then

$$\frac{1}{L}\int_M K\sqrt{g}dx^1 \wedge dx^2 + \frac{1}{L}\int_\gamma \frac{k_N^{(N)}(t)}{\sigma(t)}dt = \mathcal{X}(M),$$

where L is the Riemannian length of the indicatrix of (M, F), $k_N^{(N)}$ the signed curvature of γ over N, the scalar $\sigma = \sqrt{g_N(T, T)}$, and $\mathcal{X}(M)$ the Euler characteristic of M.

N–extremal curves

For a natural parametrized smooth curve $\gamma : [0,1] \to M$ on a Landsberg surface (M, F) with velocity field $T(t)$ we can construct the **normal vector field** $N = N(t)$ s.t.

$$g_N(N,N) = 1,\ g_N(N,T) = 0,\ g_N(T,T) = \sigma^2(t).$$

By derivation of those along γ we obtain

$$\begin{aligned} g_N(D_T^{(N)}N, N) &= 0, \\ g_N(D_T^{(N)}N, T) + g_N(N, D_T^{(N)}T) &= 0, \\ g_N(D_T^{(N)}T, T) &= \frac{1}{2}\frac{d\sigma^2(t)}{dt}. \end{aligned}$$

For **a Riemannian surface** the following are equivalent
(1) the geodesic curvature $k_N(t) = g(D_T T, N)$ vanishes
(2) the velocity vector field T is parallel transported along γ
(3) the normal vector field N is parallel transported along γ
(4) γ is a geodesic.

For **a Landsberg surface** the following are equivalent
(1) the geodesic curvature $k_N^{(T)}(t)$ vanishes
(2) the normal vector field N is parallel transported along γ,
i.e. $D_T^{(N)} N = 0$.
These curves will be called N**–extremals** of the surface M.

If γ is an N-extremal, then we have

$$g_N(N, D_T^{(N)}T) = 0,$$
$$g_N(D_T^{(N)}T, T) = \frac{1}{2}\frac{d\sigma^2(t)}{dt}.$$

It follows that the N-extremals are characterized by

$$\begin{aligned} D_T^{(N)}N &= 0 \\ D_T^{(N)}T &= \frac{d}{dt}[\log \sigma(t)]T. \end{aligned}$$

Let us construct the N-lift $\hat{\gamma}$ of an N-extremal γ to $\widetilde{TM}$ as above, i.e. $\gamma\hat{(t)} = (\gamma(t), N(t))$. The local equations of $\hat{\gamma}$ are

$$\frac{dx^i}{dt} = T^i(t) \tag{19}$$

$$\frac{dN^i}{dt} = -T^j(t) N^k \Gamma^i_{jk}(x, N), \tag{20}$$

where T is given by $g_N(N, T) = 0$.

For the initial conditions (x_0^i, N_0^i) the above equations have unique solution. Here, for (x_0^i, N_0^i) the initial velocity T_0^i is given by $g_{N_0}(N_0, T_0) = 0$.

The N-extremals are the integral lines of the vector field

$$\hat{T} = T^i(t) \frac{\delta}{\delta x^i}_{|(x,N)} \in T_N(T_x M)$$

that plays the role of the **spray** for N-extremals.

In the Riemannian case, the N-extremals coincide to the Riemannian geodesics.

In the Finslerian case, the N-extremals do not coincide with the Finslerian geodesics, nor with the geodesics of the Riemannian metric g_N.

Remark 1. For a Lansberg surface whose smooth boundary $\partial M = \gamma$ is a N-extremal, the Gauss-Bonnet formula reads

$$\frac{1}{L}\int_N K\sqrt{g}dx^1 \wedge dx^2 = \mathcal{X}(M),$$

where L is the Riemanian length of the indicatrix and $\mathcal{X}(M)$ the Euler characteristic of M.

Remark 2. Consider the case of the Minkowski plane $(\mathbb{R}^2, F)$, and a domain $D \subset \mathbb{R}^2$ bounded by the natural parametrized indicatrix curve $S = \gamma$. In this case, the Gauss-Bonnet formula implies:

$$\int_S \frac{k_N^{(N)}(t)}{\sigma(t)} dt = L,$$

where $k_N^{(N)}$ is the geodesic curvature of the indicatrix, and L the Riemannian length of the indicatrix.

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